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On an Inequality of Lyapunov for Disfocality

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1. INTRODUCTION

The Lyapunov inequality [4, Corollary 5.1] states that if $y(\cdot)$ is a non-trivial solution of

$$y''(t) + q(y)\, y(t) = 0 \tag{1.1}$$

on an interval containing the points a and b $(a < b)$ which is such that $y(a) = y(b) = 0$ then

$$\int_a^b |q(t)|\, dt > 4/(b-a). \tag{1.2}$$

It is supposed here and throughout this paper that q is a real-valued member of L^1_{loc}. The constant, 4, of (1.2) is the best possible in the sense that it cannot be replaced by a smaller one. On the other hand, it is possible to strengthen Lyapunov's inequality, see [4], in that $|q(t)|$ may be replaced by $q_+(t) := \max(0, q(t))$.

For a further means of strengthening Lyapunov's inequality it is helpful to introduce the idea of disfocality. It is well known that between any two zeros of a solution, y, of (1.1) there is a zero of y'. We may thus decompose the interval (a, b) between zeros of y into the union of the intervals (a, ξ) and $[\xi, b)$, where $y'(\xi) = 0$. It is possible now to construct inequalities similar to (1.2) on the intervals (a, ξ) and $[\xi, b)$ separately. We follow Kwong [5] and say that (1.1) is right disfocal on the interval $[\alpha, \beta)$ if the solution of (1.1) with $y'(\alpha) = 0$ has no zeros in $[\alpha, \beta)$. Left disfocality is defined in a similar way. It is clear that the absence of zeros of solutions of (1.1) on an interval (α, β) is equivalent to the right disfocality on $[\xi, \beta)$ and the left disfocality on (α, ξ). This approach has been employed by Kwong in [5] to extend Lyapunov's inequality in two directions. In this paper we look at further extensions.

Kwong's first result may be stated as follows.

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THEOREM A. *If y is a solution of* (1.1) *with $y'(0)=0$ and $y(c)=0$ then*

$$\int_0^c \int_0^t q_+(r)\,dr\,dt > 1. \tag{1.3}$$

This result may be paraphrased to state that if the inequality of (1.3) *is violated then* (1.1) *is right disfocal on* $[0, c)$.

Remark 1. There is no loss of generality in considering $[0, c]$ rather than a general interval.

In Section 3 we show, by means of a trivial observation, how Theorem A may be iterated and investigate how the iterated result compares with Theorem A.

Kwong's main result uses an inequality for integral equations to derive a result which neither implies nor is implied by Theorem A. This result has the useful feature that it involves the positive and negative parts of $q(t)$. It is reasonable to suppose that if $q(t)$ takes both negative and positive values then a keener result than Theorem A may be derived. Kwong's main result is somewhat complicated and so we refer to [5] for a full statement.

In Section 4 we derive our main result which also uses both positive and negative parts of $q(t)$. We now state this result.

THEOREM 1. *Let $\gamma(\cdot)$ denote a function with the properties*

(i) $\gamma(0)=0$

(ii) $\gamma(\cdot)$ *is differentiable on* $[0, c]$.

Set

$$Q(t) := q(t) - \gamma(t) + \gamma(t)^2 \text{ and}$$

$$A(c) := \sup_{0 \leqslant x \leqslant c} \left| \int_0^x \exp\left\{2 \int_t^x \gamma(s)\,ds\right\} Q(t)\,dt \right|$$

$$B(c) := \sup_{0 \leqslant x \leqslant c} \int_0^x \exp\left\{2 \int_t^x \gamma(s)\,ds\right\} dt.$$

If $4A(c)B(c) < 1$ then (1.1) *is right disfocal on* $[0, c)$.

2. CONSEQUENCES OF THEOREM 1

We give some corollaries to Theorem 1 which result from particular choices of γ.

COROLLARY 1. *If* $4c \sup_{0 \leqslant x \leqslant c} |\int_0^x q(t)\,dt| < 1$ *then* (1.1) *is right disfocal on* $[0, c)$.

Proof. We set $\gamma(t) := 0$ for $t \in [0, c)$ in Theorem 1.

COROLLARY 2. *If*

$$B(c) = \sup_{0 \leqslant x \leqslant c} \int_0^c e^{2\int_t^x \int_0^s q(r)\,dr\,ds}\,dt$$

and

$$A(c) = \sup_{0 \leqslant x \leqslant c} \int_0^x e^{2\int_t^x \int_0^s q(r)\,dr\,ds} \left(\int_0^t q(s)\,ds\right)^2 dt$$

then (1.1) *is right disfocal on* $[0, c)$ *if* $4A(c)B(c) < 1$.

Proof. We set $\gamma(t) := \int_0^t q(s)\,ds$.

COROLLARY 3. *If* $4c \exp\{\int_0^c (\int_0^s q(r)\,dr)_+\,ds\} \int_0^c (\int_0^t q(s)\,ds)^2\,dt < 1$ *then* (1.1) *is right disfocal on* $[0, c)$.

Proof. This follows from Corollary 2.

3. AN ITERATED FORM OF THEOREM A

Let y denote a solution of (1.1) with $y'(0) = 0$ and $y(c) = 0$. We may suppose without loss of generality that c is the least positive zero of y and $y(t) > 0$ for $t \in [0, c)$. It is also sufficient by the Sturm Comparison Theorem to consider only the case $q(t) = q(t)_+$.

We integrate (1.1) between 0 and t to obtain

$$-y'(t) = \int_0^t q_+(s)\,y(s)\,ds. \tag{3.1}$$

An integration over $[0, c]$ then yields

$$\begin{aligned} y(0) &= \int_0^c \int_0^t q_+(s)\,y(s)\,ds \\ &\leqslant y(0) \int_0^c \int_0^t q_+(s)\,ds. \end{aligned} \tag{3.2}$$

This leads to Kwong's proof of Theorem A.

Suppose now that we integrate (3.1) over the interval from s to c and obtain

$$y(s) = \int_s^c \int_0^\tau q_+(r)\, y(r)\, dr\, d\tau.$$

Substitution into (3.1) now gives

$$y'(t) = \int_0^t q_+(s) \int_s^c \int_0^\tau q_+(r)\, y(r)\, dr\, d\tau\, ds,$$

and an integration over $[0, c]$ yields

$$\begin{aligned} y(0) &= \int_0^c \int_0^t q_+(s) \int_s^c \int_0^\tau q_+(r)\, y(r)\, dr\, d\tau\, ds\, dt \\ &\leqslant y(0) \int_0^c \int_0^t q_+(s) \int_s^c \int_0^\tau q_+(r)\, dr\, d\tau\, ds\, dt. \end{aligned}$$

We thus deduce that if $y'(0) = 0$ and $y(c) = 0$ then

$$1 \leqslant \int_0^c \int_0^t q_+(s) \int_s^c \int_0^\tau q_+(r)\, dr\, d\tau\, ds\, dt. \tag{3.3}$$

In order to compare (3.3) with Theorem A we let

$$\Phi(s) := \int_s^c \int_0^\tau q_+(r)\, dr\, d\tau.$$

The inequality (3.3) represents an improvement over Theorem A if $\Phi(s) < 1$. We write

$$\begin{aligned} \Phi(s) &= \int_s^c \left\{ \int_0^s q_+(r)\, dr + \int_s^\tau q_+(r)\, dr \right\} d\tau \\ &= (c - s) \int_0^s q_+(r)\, dr + \int_s^c (c - r)\, q_+(r)\, dr \\ &= \int_0^c \psi(s, r)\, q_+(r)\, dr, \end{aligned} \tag{3.4}$$

where

$$\psi(s, r) = \begin{cases} c - s & \text{if } \ 0 \leqslant r \leqslant s \\ c - r & \text{if } \ s < r \leqslant c. \end{cases}$$

We note that $0 \leqslant \psi(s, r) \leqslant c - r$, and using this upper bound in (3.4) we have

$$\Phi(s) \leqslant \int_0^c (c-r)\, q_+(r)\, dr = \int_0^c \int_0^r q_+(s)\, ds\, dr. \tag{3.5}$$

This is inconclusive since, by Theorem A, the right hand side of (3.5) is greater than 1. On the other hand, if we use the upper bound, $\psi(s, r) \leqslant c - s$, in (3.4) we deduce that

$$\Phi(s) \leqslant (c-s) \int_0^c q_+(r)\, dr,$$

which may be less than 1.

This process may be iterated and leads to the result that if $y'(0) = 0$ and $y(c) = 0$ then for any integer n,

$$\int_0^c \int_0^{t_0} q_+(t_1) \int_{t_1}^c \int_0^{t_2} q_+(t_3) \cdots \int_{t_{2n+1}}^c \int_0^{t_{2n+2}} q(t_{2n+3})\, dt_{2n+3} \cdots dt_0 \geqslant 1.$$

4. Proof of Theorem 1

Let $y(\cdot)$ denote a solution of (1.1) with $y'(0) = 0$ and $\gamma(\cdot)$ a differentiable function to be chosen later subject to

$$\gamma(0) = 0. \tag{4.1}$$

We follow the approach of [2], see also [1, 3], and use γ to derive a regularising transformation of (1.1). We write

$$r(x) := -\left(\frac{y'}{y} - \gamma\right) \tag{4.2}$$

so that by (4.1)

$$r(0) = 0 \tag{4.3}$$

and, after substitution in (1.1),

$$r' = Q + 2\gamma r + r^2, \tag{4.4}$$

where $Q := q - \gamma' + \gamma^2$. We rearrange (4.4) as

$$r' - 2\gamma r = Q + r^2$$

and integration yields, from (4.3),

$$r(x) = \int_0^x e^{2\int_t^x \gamma(s)\, ds} Q(t)\, dt + \int_0^x e^{2\int_t^x \gamma(s)\, ds} r(t)^2\, dt. \tag{4.5}$$

Let

$$A(X) := \sup_{0 \leqslant x \leqslant X} \left| \int_0^x e^{2\int_t^x \gamma(s)\,ds} Q(t)\,dt \right|$$

$$B(X) := \sup_{0 \leqslant x \leqslant X} \int_0^x e^{2\int_t^x \gamma(s)\,ds}\,dt$$

$$R(X) := \sup_{0 \leqslant x \leqslant X} |r(x)|.$$

It is clear from (4.5) that

$$|r(x)| \leqslant A(X) + B(X)R(X)^2 \qquad \text{for} \quad x \in [0, X],$$

and thus

$$R(X) \leqslant A(X) + B(X)R(X)^2. \tag{4.6}$$

LEMMA. *If X is such that $4A(X)B(X) < 1$ then*

$$R(x) < 2A(x) \qquad \textit{for} \quad x \in [0, X].$$

Proof. We know that $R(0) = 0$ so if the result were false there would be a least value of x, x_0, say, for which $R(x_0) = 2A(x_0)$; thus, from (4.6),

$$\begin{aligned} 2A(x_0) &\leqslant A(x_0) + B(x_0)R(x_0)^2 \\ &= A(x_0)(1 + 4A(x_0)B(x_0)) \end{aligned}$$

which gives a contradiction.

In particular the Lemma shows that if $4A(c)B(c) < 1$ then

$$\left| \frac{y'(x)}{y(x)} - \gamma(x) \right| \leqslant 2A(c), \qquad x \in [0, c].$$

Thus, if $\gamma(\cdot)$ is bounded for $x \in [0, c]$ then y has no zeros in $[0, c]$.

REFERENCES

1. F. V. ATKINSON, Asymptotics of an eigenvalue problem with an interior singularity, preprint.
2. B. J. HARRIS, On the oscillation of solutions of linear differential equations, *Mathematika* **31** (1984), 214–226.
3. B. J. HARRIS, A note on a paper of Atkinson concerning the asymptotics of an eigenvalue problem with interior singularity, *Proc. Roy. Soc. Edinburgh A* **110** (1988), 63–71.
4. P. HARTMAN, "Ordinary Differential Equations," Wiley, New York, 1964.
5. M. K. KWONG, On Lyapunov's inequality for disfocality, *J. Math. Anal. Appl.* **83** (1981), 486–494.